ON THE PROBLEM OF ATTENUATION OF A LINEAR SYSTEM

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1. Consider a controlled system characterized by Eq.

$$dx/dt = Ax + bu \tag{1.1}$$

where x is an n-vector in the phase coordinates x_i; u is a scalar control; A is an $(n \times n)$-constant matrix, and b is a constant n-vector. The problem consists of choosing a control $u(t)\,(t_\alpha \leqslant t \leqslant t_\beta)$ which will bring system (1.1) from the state $x(t_\alpha) = x^\alpha$ to a position of equilibrium $x(t_\beta) = 0$. A great deal of attention has been devoted to this well known problem (see, for example, [1 to 6]). Herein, the investigation is conducted on the assumption that the controls u are represented by generalized functions. This choice of class for the allowable actions u, naturally places it within the class of control problems utilizing the approach proposed in [6]. Indeed, if we initially let $u(t)$ be an integrable function, we can write, with the aid of Cauchy's formula [7]

$$x(t_\beta) = X[t_\beta, t_\alpha]\, x^\alpha + \int_{t_\alpha}^{t_\beta} H[t_\beta, \tau]\, u(\tau)\, d\tau \tag{1.2}$$

where $X[t, t_0] = \exp A(t - t_0) = \{x_{ij}(t, t_0)\}$ is the fundamental matrix of system (1.1) (for $u(t) \equiv 0$),

$$H[t_\beta, \tau] = \{h^{(i)}(\tau)\} = X[t_\beta, \tau]\, b$$

It is convenient to treat the components of the second term in the right-hand side of (1.2) as values of a linear functional $\varphi_u[h^{(i)}]$ on the functions $h^{(i)}(\tau)$. Hence, (1.2) may be written in terms of the coordinates as follows:

$$x_i(t_\beta) = \sum_{j=1}^{n} x_{ij}(t_\beta, t_\alpha)\, x_j^\alpha + \varphi_u[h^{(i)}(\tau)] \qquad (i = 1, \ldots, n) \tag{1.3}$$

The functions $h^{(i)}(\tau)\,(t_\alpha \leqslant \tau \leqslant t_\beta)$ may be continued as functions defined for $-\infty < \tau < \infty$ possessing derivatives of all orders and vanishing for $|\tau| > \zeta$, where ζ is sufficiently small. Then these functions can be considered as elements of a linear space $K_\zeta\{h\}$ utilized in the construction of generalized functions [8]. Now, making use of (1.3), the allowable controls u may be extended to the class of generalized functions whose support lies entirely on the segment $[t_\alpha, t_\beta]$. In this definition, we assume that the generalized control u will bring the system (1.1) to the state $x(t_\beta) = \{x_i(t_\beta)\}$ (1.3), where φ_u is a linear functional on $K_\zeta\{h\}$ embodying the given effect. Particular cases of such a situation have been investigated in [3, 5 and 9], which, in fact, considered the simplest generalized controls including impulsive effects.

Formulation of the equilibrium problem on the basis of generalized disturbances and employing generalized function theory has been presented in [10 and 11]. The present work is concerned with two problems: 1) The attenuation of system (1.1) by means of generalized controls u are composed of δ-functions and their derivatives $\delta^{(1)}, \delta^{(2)}, \ldots \delta^{(n-1)}$, and 2) Optimum attenuation of system (1.1) under minimum conditions of a specified intensity $\varkappa[u]$ of the generalized action u.

2. The controls u, attenuating (1.1), will be sought in the form

$$u = \lambda_1\delta(t-\vartheta) + \ldots + \lambda_n\delta^{(n-1)}(t-\vartheta) \tag{2.1}$$

where ϑ is some fixed moment in time within the interval $[t_\alpha, t_\beta]$ and λ_i are the desired constants. From the definition of the functions $\delta^{(k)}(t)$ and from the known properties of the matrix $X[t_\beta, \tau]$, we have

$$\int_{t_\alpha}^{t_\beta} X[t_\beta, \tau]\, b\delta^{(k)}(\tau-\vartheta)\,d\tau = (-1)^k\left(\frac{d^k X[t_\beta, \tau]}{d\tau^k}\right)_{\tau=\vartheta} b = X[t_\beta, \vartheta]\, A^k b$$

Then, substituting for u from (2.1) into (1.2) wherein the right-hand side behaves in accordance with Section 1, and setting $x(t_\beta) = 0$, we obtain the following Eq. for the unknown λ_i:

$$-X[t_\beta, t_\alpha]\, x^\alpha = X[t_\beta, \vartheta](\lambda_1 b + \lambda_2 Ab + \lambda_3 A^2 b + \ldots + \lambda_n A^{n-1} b) \tag{2.2}$$

Eq. (2.2) has a solution for arbitrary initial conditions if and only if the vectors b, Ab, ..., $A^{n-1}b$ are linearly independent, i.e. if the condition for a general state [1] is satisfied or, in other words, if the condition for complete control [4] of system (1.1) is satisfied. In particular, for $\vartheta = t_\alpha$, (2.2) takes the form

$$-x^\alpha = \lambda_1 b + \lambda_2 Ab + \lambda_3 A^2 b + \ldots + \lambda_n A^{n-1} b \tag{2.3}$$

Thus we conclude that a generalized control u of the form (2.1) which will result in the attenuation of system (1.1) exists if and only if the vector $x = x^\alpha$ lies in the subspace generated by the vectors $b, Ab, \ldots, A^{n-1}b$; if $\vartheta = t_\alpha$, then the λ_i are the coordinate components of the x^α vector upon its resolution along the $A^{i-1}b$ vectors.

This conclusion corresponds, of course, to a known fact in the theory of controls of linear systems whose action is described by generalized functions $u(t)$.

3. Suppose now that x is an element in some infinite-dimensional linear space $\{x\}$. Suppose, moreover, that in (1.1) u is again a scalar, b is an element of $\{x\}$ and A denotes a linear operator for which Cauchy's Formula (1.2) holds, whereupon the function

$$\exp A(t-t_0) = \sum_{i=0}^{\infty} \frac{A^i (t-t_0)^i}{i!}$$

has the known regularity properties of group operators [12]. Suppose further that the elements $y^{[k]} = A^{k-1}b$ ($k = 1, 2, \ldots$) constitute a basis in the $\{x\}$ space. Consequently, each element x in $\{x\}$ may be represented in the form

$$x = \sum_{i=1}^{\infty} \xi_i y^{[i]} = \sum_{i=1}^{\infty} \xi A^{i-1} b \tag{3.1}$$

Assume that the norm $\rho[x]$ which defines the metric in the $\{x\}$ space is chosen so as to satisfy the condition: for every element x (3.1) having a finite norm $\rho[x] < \infty$ and for all $\varepsilon > 0$ there exists an N such that

$$\rho\left[\sum_{i=N+1}^{\infty} \xi_i y^{[i]}\right] < \varepsilon$$

Then for any initial condition $x = x^\alpha$ having a finite norm and for all $\varepsilon > 0$ we may construct a generalized control u of the form (2.1) which will bring (1.1) to a state $x = x(t_\beta)$ satisfying the condition $\rho[x(t_\beta)] < \varepsilon$. In order to prove the preceding statement it is sufficient, in (2.1), to choose $\vartheta = t_\alpha$, $n = N$ and to set $\lambda_i = -\xi_i$ ($i = 1, \ldots, N$). Then repeating the developments of Section 2 which, by assumption, hold for the operator A, we find that the specified control u will bring the object system to a state

$$x(t_\beta) = \sum_{i=N+1}^{\infty} \xi_i y^{[i]}$$

Thus the above statement has been proved.

4. The control u which was investigated formally in Sections 2 and 3 causes attenuation in system (1.1) (to a state of equilibrium or to a state lying in an ε-neighborhood of the equilibrium point $x = 0$, respectively) instantly at $t = \vartheta$. Such a control is, of course, a practical impossibility. However, such a control may be usefully investigated in those

cases in which the attenuation is achieved through effects which are relative to a system such as (1.1) are of high intensity and short time duration $[t_\alpha, t_\alpha + \varepsilon]$. Then controls are realized in practice, and the δ-functions and their derivatives $\delta^{(k)}$ are used as approximations of ordinary functions $\Delta_\varepsilon^{(k)}(t)$:

$$\Delta_\varepsilon^{(0)}(t) = 1 / \varepsilon \quad \text{for} \quad 0 \leqslant t < \varepsilon, \quad \Delta_\varepsilon^{(0)}(t) = 0 \quad \text{for other} \quad t$$

$$\Delta_\varepsilon^{(1)}(t) = 4 / \varepsilon^2 \quad \text{for} \quad 0 \leqslant t < \varepsilon / 2,$$

$$\Delta_\varepsilon^{(1)}(t) = -4 / \varepsilon^2 \quad \text{for} \quad \varepsilon / 2 \leqslant t \leqslant \varepsilon$$

$$\Delta_\varepsilon^{(1)}(t) = 0 \quad \text{for other} \quad t \quad \text{etc.}$$

As $\varepsilon \to 0$, the result of the control action

$$u_\varepsilon(t) = \sum_{i=1}^{n} \lambda_i \Delta_\varepsilon^{(i-1)}(t - \vartheta) \quad (t_\alpha \leqslant t \leqslant t_\alpha + \varepsilon) \tag{4.1}$$

approaches the value $x(t_\beta)$ which is obtained by formal utilization of the control u (2.1). This circumstance also determines the real sense of the last generalized equation. Note also that in this manner utilization of the control u (2.1) and the associated controls $u_\varepsilon(t)$ (4.1) permits the estimation of the order of growth of the controlling effects $u_\varepsilon(t)$ as $\varepsilon \to 0$. This order is generally of the order $1/\varepsilon^n$.

5. Consider the attenuation problem for system (1.1) subject to a minimum condition of intensity $\varkappa[u]$ for the generalized control u. By the very definition of the generalized controls u which are here under consideration, their intensity would naturally be evaluated from the results of the corresponding operations $\varphi_u[h]$ on the elements $h(\tau)$ in the $K_\zeta[h(\tau)]$ space. Hence, consider some function (more precisely a functional) $\rho_\mu[h(\tau)]$ (The subscript μ is a number), and assume for definiteness that $\varkappa[u] \leqslant \mu$ if and only if the inequality

$$\varphi_u[h(\tau)] \leqslant \rho_\mu[h(\tau)] \tag{5.1}$$

is satisfied for all $h(\tau)$ in $K_\zeta\{h(\tau)\}$. Clearly, this estimate of the intensity $\varkappa[u]$ is an automatic extension to the generalized controls u of the same treatment of intensity $\varkappa[u]$ as that which was given to the intensity $\varkappa[u]$ of ordinary controls $u(t)$ in [6 and 9], wherein they set $\varkappa[u] = \rho^*[u(\tau)]$ and $\rho_\mu[h(\tau)] = \mu\rho[h(\tau)]$ $(\mu > 0)$, where the symbols ρ and ρ^* are the norms in a suitable function space $B\{h(\tau)\}$ and in the adjoint space $B^*\{u(\tau)\}$, respectively. Then it becomes clear that here also the controls problem may be conveniently treated as a problem of moments, and, for suitably behaved functions $\rho_\mu[h]$, the solution of this problem, and from it the solution for the controls, is obtained as a corollary to the Hahn-Banach theorem on the extension of a linear functional [13]. Hereafter, in accordance with the above discussion, we will assume that the function ρ_μ satisfies two conditions: 1) $\rho_\mu[h(\tau)]$ depends only on the values of the functions $h(\tau)$ which are admissible for $t_\alpha \leqslant \tau \leqslant t_\beta$; 2) for any $h(\tau)$ and $g(\tau)$ in K_ζ and for arbitrary values of $\alpha \geqslant 0$, the following conditions hold:

$$\rho_\mu[h + g] \leqslant \rho_\mu[h] + \rho_\mu[g], \quad \rho_\mu[\alpha h] = \alpha\rho_\mu[h] \tag{5.2}$$

Then indeed the solution of the problem of attenuation of system (1.1) follows the known procedure for solving a moments problem. We will describe it here briefly for completeness of the development.

Setting $x_i(t_\beta) = 0$ $(i = 1, \ldots, n)$ in (1.3), we obtain Eqs.

$$\varphi_u[h^{(i)}(\tau)] = c_i \quad (i = 1, \ldots, n) \qquad (c = \{c_i\} = -X[t_\beta, t_\alpha]x^\alpha) \tag{5.3}$$

Consider, within the $K_\zeta\{h\}$ space, a subspace $K_n\{h\}$ consisting of all those functions $h(\tau)$ in K_ζ which, for $t_\alpha \leqslant \tau \leqslant t_\beta$, have the form

$$h(\tau) = l_1 h^{(1)}(\tau) + \ldots + l_n h^{(n)}(\tau) \tag{5.4}$$

and define, for these functions, a linear operation $\varphi_n[h]$, such that

$$\varphi_n[h] = l_1 c_1 + \ldots + l_n c_n \tag{5.5}$$

Here l_i are any real numbers. The operation φ_n (5.5) may be performed if and only if the quantity $\nu = l_1 c_1 + \ldots + l_n c_n$ vanishes every time that

$$l_1 h_1(\tau) + \ldots + l_n h_n(\tau) \equiv 0 \qquad \text{for} \quad t_\alpha \leqslant \tau \leqslant t_\beta$$

If the above condition is not fulfilled, then the moments problem (5.3), and therefore the controls problem, has no solution. Thus, suppose that the operation (5.5) is meaningful. This operation satisfies (5.3), but it is defined only on K_n. Let us examine the possibility of extending φ_n to the operations φ_u, defined over the entire K_ζ space. Choose some value of μ.

If there exist values of l_i for which the inequality

$$\varphi_n[h] = l_1c_1 + \ldots + l_nc_n > \rho_\mu\ [h] \tag{5.6}$$

holds, then the moments problem (5.3) again has no solution for $\varkappa\ [u] \leqslant \mu$. However, suppose that the inequality

$$\varphi_n[h] = \sum_{i=1}^{n} l_ic_i \leqslant \rho_\mu[h] \tag{5.7}$$

holds for a given μ for all values of l_i.

Then, in accordance with the previously cited Hahn-Banach Theorem [13], the operation $\varphi_n\ [h]$ may be extended to the functional $\varphi_u\ [h]$, which is defined over the entire $K_\zeta\{h\}$ space and satisfying (5.3) as well as the inequality $\varphi_u\ [h] \leqslant \rho_\mu\ [h]$ for all $h(\tau)$ in K_ζ. In view of the fact that every function $h(\tau)$ in K_ζ which satisfies the condition $h(\tau) \equiv 0$ for $t_\alpha \leqslant \tau \leqslant t_\beta$ also satisfies $\varphi_u\ [h]\ = 0$, by construction, we conclude that the support of u is indeed contained within the segment $[t_\alpha, t_\beta]$. Hence, when the inequality (5.7) is satisfied, there exists a generalized control u causing the attenuation of system (1.1) and having an intensity $\varkappa\ [u] \leqslant \mu$.

We will now make use of the known relation

$$\sum_{i=1}^{n} l_ih^{(i)}(\tau) = l'H[t_\beta, \tau] = l'X[t_\beta, \tau]\, b = b'S[\tau, t_\beta]\, l \tag{5.8}$$

where the prime denotes the transpose of a quantity; the symbol $S[t, t_0]$ represents the fundamental matrix of the system whose motion is given by Eq.

$$\frac{ds}{dt} = -A_s' \tag{5.9}$$

which is adjoint to Eq. $dx/dt = Ax$, corresponding to the original system. Then the considerations given above in connection with the inequalities (5.6) and (5.7) may be summarized in the following rule determining the solution to the pro blem of optimal attenuation for system (1.1) and in a similar, known rule for generalized controls [6 and 14].

Theorem 5.1. In order to solve the optimal attenuation problem for system (1.1), it is necessary to examine the motion $s(t) = S[t, t_\beta]\, s(t_\beta)$ for the adjoint system (5.9) and to construct the quantity

$$\gamma_\mu\ [s(t_\beta)] = \rho_\mu\ [b'S\ [\tau,\ t_\beta] s(t_\beta)]$$

Let μ° be the smallest value of μ for which the inequality

$$\gamma_\mu\, [s(t_\beta)] - s'(t_\beta)\, c \geqslant 0 \tag{5.10}$$

is satisfied for all $s[t_\beta]$, whereupon

$$\min_s\, [\gamma_{\mu^\circ}\, [s(t_\beta)] - s'(t_\beta)\, c] = 0 \tag{5.11}$$

for $\|s(t_\beta)\| = 1$. Then the desired min $\varkappa\ [u]\ = \mu^\circ$ and the optimal control u° with intensity $\varkappa\ [u^\circ]\ = \mu^\circ$ satisfies the maximum condition

$$\varphi_{u^\circ}[b'S[\tau, t_\beta]\, s^\circ] = \max_u \quad \text{for} \quad \varkappa[u] \leqslant \mu^\circ \tag{5.12}$$

Here $s^\circ = s^\circ(t_\beta)$ is the solution to problem (5.11) and the symbol $\|s\|$ denotes the Euclidean norm of the vector s.

Note. The supporting theory of generalized functions assumes that the functionals φ_u which embody these functions are continuous in the countably normed space K. In this connection, it must be assumed that, by the choice of $\rho\mu[h]$, the condition $\varphi_u\ [h] \leqslant \rho_\mu\ [h]$ guarantees continuity of φ_u.

It is frequently convenient to transform the moments problem (5.3) by premultiplying initially both sides of (1.2) by $X^{-1}[t_\beta, t_\alpha]$. Then we obtain a rule for the solution of the problem which is similar to Theorem 5.1, but in that case it is necessary to determine the smallest value $\mu = \mu^\circ$ not from (5.11) but from the condition

$$\min_s\, [\gamma_{\mu^\circ}\, [s(t_\alpha)] - s'(t_\alpha)\, c] = 0 \tag{5.13}$$

for $\|s(t_\alpha)\| = 1$ and $c = -x^\alpha$. Here $\gamma_\mu[s(t_\alpha)] = \rho_\mu[b'S[\tau, t_\alpha] s(t_\alpha)]$. The optimal control u° is determined again from the maximum condition

$$\varphi_{u^\circ}[b'S[\tau, t_\alpha] s^\circ] = \max_u \quad \text{for} \quad \varkappa[u] \leqslant \mu^\circ \tag{5.14}$$

similarly to (5.12), but where $s^\circ = s^\circ(t_\alpha)$ is the solution of (5.13). In view of the homogeneity of $\gamma_\mu[s]$ and $\nu = s'c$ with respect to s, the condition $\|s\| = 1$ in problems (5.11) and (5.13) could be chosen arbitrarily and may be replaced by any other similar condition.

Theorem 5.1. is concerned with bringing system (1.1) to a state of equilibrium $x(t_\beta) = x^\beta = 0$. If it is desired to bring the system (1.1) to one of the states $x(t_\beta) = x^\beta$ which compose a given manifold $M\{x^\beta\}$, then, as in the case of ordinary controls [14], the problem can be solved as a moments problem (5.3) utilizing Theorem 5.1 with

$$c[x^\beta] = x^\beta - X[t_\beta, t_\alpha] x^\alpha$$

and minimizing $\mu^\circ[x^\beta]$ for the x^β in M.

If the manifold M is a convex, compact set in the $\{x\}$ space, then, as in the case of ordinary controls [14], the operations involving the minimum with respect to s and the maximum with respect to x^β are interchanged, resulting in a known situation in game theory, with a saddle point [15], and having a real geometrical meaning if the relations determining the solution are interpreted as partition conditions for a convex set M and a convex accessible region G for the process (1.1) in the $\{x\}$ space (cf., for example [16 and 17]).

For the function $\rho_\mu[h]$, it is convenient to choose quantities depending on $\eta^{(k)} = \max_\tau |d^k h/d\tau^k|$. In particular, choosing $\rho_\mu[h] = \mu \max_\tau |h(\tau)|$, we obtain controls consisting of impulse δ-functions which have been investigated, for example, in [3, 5 and 9]. Generalized functions of higher order are then excluded as a result of the specified inequality $\varphi_u[h] \leqslant \mu \max_\tau |h(\tau)|$, which does not take into account derivatives of $h(\tau)$. Setting $\rho_\mu[h] = \mu \max[\eta^0, \eta^{(k)}]$, we obtain a wider class of generalized functions u which will include functions $\delta^{(1)}(t - \vartheta) = d\delta(t - \vartheta)/dt$, etc.

6. *Examples*. Consider the controlled motion of a material point along the line ξ, the motion being described by Eq.

$$\frac{d^2\xi}{dt^2} + \omega^2\xi = u \qquad (\omega \geqslant 0) \tag{6.1}$$

and it is necessary to bring this point in the time $t_\alpha = 0 \leqslant t \leqslant t_\beta = \pi/\omega$ from the state

$$\xi(0) = \xi_0, \qquad (d\xi(t)/dt)_{t=0} = \xi_1$$

to the equilibrium state

$$\xi(\pi/\omega) = 0, \qquad (d\xi(t)/dt)_{t=\pi/\omega} = 0.$$

For the minimized intensity $\varkappa[u]$, we choose the quantity defined by $\rho_\mu[h] = \mu \max_\tau [|h(\tau)|, |dh(\tau)/d\tau|]$. Then, replacing (6.1) by the system

$$\frac{dx_1}{dt} = x_2, \qquad \frac{dx_2}{dt} = -\omega^2 x_1 + u \tag{6.2}$$

and applying to this system Theorem 5.1, we find that for $\omega > 1$, the minimum of the quantity

$$\gamma_{\mu^\circ}[s(t_\beta)] = \rho_{\mu^\circ}[b'S[\tau, t_\beta] s(t_\beta)] = \mu^\circ \max_\tau [|b's(\tau)|, |db's(\tau)/d\tau|]$$

is defined by the term $\max_\tau |db's(\tau)/d\tau|$, while for $\omega < 1$ this minimum is defined by the term $\max_\tau |b's(\tau)|$ $s(\tau) = S[\tau, t_\beta] s(t_\beta)$. Hence, in accordance with the maximum condition (5.11), we conclude that, for $\omega \leqslant 1$, the optimal control u° must be chosen in the form of an impulse $u^\circ(t) = \lambda\delta(t - \vartheta)$ which instantly stops the point at that instant $t = \vartheta$ when it passes through the equilibrium position $\xi = 0$. On the other hand, for $\omega > 1$, for the chosen value of the intensity $\varkappa[u]$, the optimal control must be chosen in the form $u^\circ(t) = \lambda\delta^{(1)}(t - \vartheta)$, which is applied at that instant $t = \vartheta$ at which the velocity $d\xi(t)/dt$ of the moving point is zero. The given effect may be considered as two oppositely direct extraordinarily strong impulses, with one immediately following the other. The first impulse imparts to the point ξ an enormous velocity transfering it to the position $\xi = 0$ while the second impulse immediately dampens this velocity at the point $\xi = 0$. We note here that in this example the optimal control u° has exactly the form of the generalized control which was investigated in Section 2 to 4.

As a second example, we can examine the attenuation problem for a linear system with aftereffect [18 and 19] after a time $T = 2\tau$. Here, for example, for $n = 2$ the control $u(t)$ for

$0 < t \leqslant T$ will again be determined by the condition for retaining the trajectory $x(t)$ with the required subspaces and at the instant $t = +0$ a control may be sought whose form is a linear combination $u = \gamma_1 \delta(t) + \gamma_2 \delta'(t)$. We note here that in a note [18] it was erroneously stated that the problem there under consideration always has a solution when the vectors b and Ab satisfy the condition for a general state. Indeed, in some degenerate cases (when $h_2(\vartheta) \equiv 0$) this condition is not sufficient; for example, when

$$A = \begin{pmatrix} 0, & 1 \\ 0, & 0 \end{pmatrix}, \quad b = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad g_{12} = -1$$

Thus, we are reminded that the sufficient conditions must fulfill a requirement guaranteeing the inequality $h_2(\vartheta) \not\equiv 0$. The authors were informed by Kurzhanskii that a similar inaccuracy exists in paper [19].

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